A special class of pure O-sequences

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Abstract

Let $x_1, \ldots, x_s$ represent distinct indeterminates with $\deg x_i = 1$, for $i = 1, \ldots, s$. A nonempty, finite set $\mathcal{A}$ of monomials in $x_1, \ldots, x_s$ is called an *order ideal of monomials* if for any $u \in \mathcal{A}$ and any monomial v that divides u, we have $v \in \mathcal{A}$. In particular, $1 \in \mathcal{A}$ for any order ideal of monomials $\mathcal{A}$. We say that $\mathcal{A}$ is *pure* if the maximal elements of $\mathcal{A}$, with respect to divisibility, all have the same degree. The *h-vector* of $\mathcal{A}$ is defined as $h(\mathcal{A}) = (h_0, h_1, \ldots, h_n)$, where

$$n = \max\{\deg u : u \in \mathcal{A}\} \text{ and } h_i = \left|\{u \in \mathcal{A} : \deg u = i\}\right|, \text{ for } 0 \leq i \leq n.$$

Clearly, $h_0 = 1$. A finite sequence of positive integers $h = (h_0, h_1, \ldots, h_n)$ is called an *O-sequence* if there exists an order ideal of monomials $\mathcal{A}$ with $h = h(\mathcal{A})$. An O-sequence h is *pure* if there exists a pure order ideal of monomials $\mathcal{A}$ with $h = h(\mathcal{A})$.

A classification of the possible O-sequences is essentially due to Macaulay. On the other hand, an explicit characterization of pure O-sequences seems entirely out of reach, despite much effort by many researchers. The purpose of the present talk is to classify the pure O-sequences of the form $(1, a, a, \ldots, a, b)$. This is a joint work with Tài Huy Hà and Fabrizio Zanello (arXiv:2404.08183).